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String equations for the KdV hierarchy and the Grassmannian*

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Abstract. The Grassmannian model for the Korteweg–de Vries hierarchy is used to describe the translational, Galilean and scaling self-similar solutions of this hierarchy. These solutions are characterized by the string equations appearing in two-dimensional quantum gravity. In particular, the subsets of the Sato Grassmannian corresponding to solutions of the string equations are found. The well known Adler–Moser rational solutions are obtained as well as a three-parameter family of solutions associated with the Painlevé II equation.

1. Introduction

Recently, there has been an increasing interest in the study of solutions of the Korteweg–de Vries (KdV) hierarchy which satisfy the so-called string equation. This is motivated by the papers [7, 4, 13] where it is shown that the double scaling limit of a Hermitian matrix model gives a model for two-dimensional quantum gravity where the dependence on the parameters of the specific heat is given by the KdV hierarchy and a string equation. In [18], the Grassmannian model of Sato [27] for the KdV flows is used to give a geometrical picture for the solutions of the string equation, which is also considered in [21] where one can find a more analytical treatment of the subject based on the zero-curvature condition.

In the papers [19, 20] it was noticed that the string equation corresponds to Galilean self-similar solutions of the KdV hierarchy (see also [15]). Motivated by the anomalous behaviour of the solutions of the string equation a non-perturbative approach to 2D quantum gravity, it was proposed in [6, 3, 16] described by an extension of the string equation that is associated with scaling self-similar solutions of the KdV hierarchy. In [5, 17] a boundary cosmological term is added to the string equation, this new equation corresponds to Galilean and scaling self-similarity of the KdV hierarchy.

It is the aim of this paper to present a Grassmannian description of the string equations. Our approach is based on the Birkhoff factorization problem for the KdV flows [14, 28]. In the second section we introduce the potential KdV hierarchy, its zero-curvature formulation and its local symmetries. The local symmetries are either isospectral, translations, or the non-isospectral, Galilean and scaling transformations. A more general string equation is found as the self-similarity condition under a local symmetry (a solution is self-similar if it remains invariant under the symmetry transformation). In the next section we introduce the factorization problem in connection with the potential KdV hierarchy, as well as the Grassmannian model. In this section we characterize which subspaces of the Grassmannian

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are connected with the solutions of the string equations. Finally, in section 4 we describe explicitly the subsets of the Sato Grassmannian associated with self-similar solutions. This description allows us to obtain the well known Adler–Moser rational solutions [2] as a particular case of scaling and translation self-similarity. Specifically for the KdV equation a three-parameter family of scaling self-similar solutions (solutions of the Painlevé II equation) is found, in agreement with the results of [9], see also [1]. We also consider the Kac–Schwarz description regarding the Galilean invariance. As a result one concludes that the existence of certain hypersurfaces in the Sato Grassmannian corresponds to a particular self-similarity condition. From this it follows, for example, that one can give in the Galilean case, i.e. for the usual string equation, a $2m$-dimensional surface in the Sato Grassmannian associated with self-similar solutions defined in a neighbourhood of $t_{2m+3} = 2/(2m+3)$. This result was found in [21] but the coordinates used were certain Stokes parameters instead of the initial data for the Gel'fand–Dikii potentials used here. The case studied in [18] corresponds to the particular choice $m = 0$.

2. Potential KdV hierarchy and string equations

The potential KdV hierarchy for a scalar function p that depends on an infinite number of variables $\boldsymbol{t} := \{t_{2n+1}\}_{n\geqslant 0}$, the local coordinates for the time manifold $\mathcal{T}$, is a collection of compatible equations

$$\partial_{2n+1} p = -2R_{n+1}[p]$$

where $\partial_{2n+1} := \partial/\partial t_{2n+1}$ and R_{n+1} are the Gel'fand–Dickii [10] coefficients for the expansion of the kernel of the resolvent of the associated Schrödinger equation with potential $u = -2\partial_1 p$. These coefficients are polynomials in u and its ∂_1 derivatives. The first equation of the hierarchy is the potential KdV equation $4\partial_3 p = \partial_1^3 p - 6(\partial_1 p)^2$ that for the potential u is the KdV equation $4\partial_3 u = \partial_1^3 u + 6u\partial_1 u$, for which Novikov [23] gave a zero-curvature representation in terms of a differential 1-form $\chi(\lambda)$ that depends on a complex spectral parameter $\lambda \in \mathbb{C}$. The KdV hierarchy has a similar formulation. Let χ be the 1-form on $\mathcal{T}$ defined by

$$\chi(\lambda) := \sum_{n\geqslant 0} L_{2n+1}(\lambda)\, \mathrm{d}t_{2n+1} \tag{2.1}$$

where

$$L_{2n+1}(\lambda) := \begin{pmatrix} -\frac{1}{2}\partial_1 \rho_n(\lambda) & \rho_n(\lambda) \\ (\lambda - u)\rho_n(\lambda) - \frac{1}{2}\partial_1^2 \rho_n(\lambda) & \frac{1}{2}\partial_1 \rho_n(\lambda) \end{pmatrix}$$

with

$$\rho_n(\lambda) = 2\sum_{m=0}^{n} \lambda^m R_{n-m}[u].$$

Then, the KdV hierarchy is equivalent to the zero-curvature condition,

$$[d - \chi, d - \chi] = 0$$

where d is the exterior derivative $d := \sum_{n\geqslant 0} \mathrm{d}t_{2n+1}\partial_{2n+1}$. The potential KdV hierarchy is equivalent to the zero-curvature condition for the 1-form

$$\omega_+ := d\phi \cdot \phi^{-1} + \mathrm{Ad}\,\phi\,\chi \tag{2.2}$$

where

$$\phi := \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}.$$

Let us now consider the symmetries defined by translations, scaling and Galilean transformations.

The infinite set of translational symmetries are the isospectral symmetries of the hierarchy in the sense that they preserve the associated spectral problem. In fact the flows in the hierarchy are defined by the generators ∂_{2n+1} of translations. If we define

$$\vartheta(t) := t + \theta$$

where

$$\theta := \{\theta_{2n+1}\}_{n\geqslant 0} \in \mathbb{C}^\infty$$

we have a local action of the Abelian group $\mathbb{C}^\infty$ over the time manifold $\mathcal{T}$. If p is a solution to the hierarchy then $\vartheta^* p$ is also a solution.

There is the non-isospectral action of the Virasoro algebra on the set of solutions to the potential KdV hierarchy [12]. The flows defined by this algebra are, in general, non-local in ∂_1, but for the Galilean and scaling transformations, which are non-isospectral symmetries, the action is local. In what follows, we shall only be interested in local symmetries.

The scaling symmetry is a natural extension of the scaling symmetry of the KdV equation. We define

$$\varsigma_w(t) := \{\mathrm{e}^{w(n+\frac{1}{2})} t_{2n+1}\}_{n\geqslant 0}$$

where $w \in \mathbb{C}$. If p is a solution of the potential KdV hierarchy then $\mathrm{e}^{w/2}\varsigma_w^* p$ is a solution as well. We have an additive action of $\mathbb{C}$ over $\mathcal{T}$.

The Galilean symmetry of the KdV equation has a less trivial extension to the whole hierarchy. We define a Galilean transformation locally as the additive action of $\mathbb{C}$ over $\mathcal{T}$ given by

$$\gamma_v(t) := \left\{ \frac{1}{\Gamma(n+\frac{3}{2})} \sum_{m=n}^{\infty} \frac{\Gamma(m+\frac{3}{2})}{(m-n)!} v^{m-n} t_{2m+1} \right\}_{n\geqslant 0}$$

this series converges if $|t_{2n+1}| < |v|^{-n}$. If p is solution of the potential hierarchy then so is $\gamma_v^* p - \frac{1}{2} v t_1$.

The related fundamental vector fields, infinitesimal generators of the action of translations, scalings and Galilean transformations are, in each case,

$$\partial_{2n+1}, n \geqslant 0 \qquad \varsigma = \sum_{n\geqslant 0} (n+\tfrac{1}{2}) t_{2n+1}\partial_{2n+1} \qquad \gamma = \sum_{n\geqslant 1} (n+\tfrac{1}{2}) t_{2n+1}\partial_{2n-1}$$

and generate the linear space $\mathbb{C}\{\partial_{2n+1}, \varsigma, \gamma\}_{n\geqslant 0}$ which is a Lie algebra with Lie brackets

$$[\partial_{2n+1}, \varsigma] = (n+\tfrac{1}{2})\partial_{2n+1} \qquad [\partial_{2n+1}, \gamma] = (n+\tfrac{1}{2})\partial_{2n-1} \qquad [\varsigma, \gamma] = \gamma.$$

A self-similar solution under any of the mentioned symmetries is a solution which remains invariant under the corresponding transformation. Consider the following vector field belonging to this Lie algebra

$$X := \vartheta + v\gamma + w\varsigma \qquad \text{with} \qquad \vartheta = \sum_{n\geqslant 0}\theta_{2n+1}\partial_{2n+1}$$

defining a superposition of translations, Galilean and scaling transformations. If p is a solution then the function

$$\mathrm{e}^{w/2}\exp(X)p - v(\sinh(w/2)/w)t_1$$

is a solution as well. Here we have used the fact

$$\exp(X) = \exp(\hat{\vartheta})\exp(2v(\sinh(\tfrac{1}{2}w)/w)\gamma)\exp(w\varsigma)$$

where, by the Campbell–Hausdorff formula [25],

$$\hat{\vartheta} := \sum_{n>0}\frac{1}{n!}(\mathrm{ad}(v\gamma + w\varsigma))^{n-1}\vartheta$$

is again a vector field corresponding to a translation. Therefore, a solution p of the potential KdV hierarchy is self-similar under translation, scaling and Galilean transformations if

$$(\theta + v\gamma + w\varsigma)p - \tfrac{1}{2}vt_1 + \tfrac{1}{2}wp = 0. \tag{2.3}$$

Let us denote

$$\mathcal{R} := \sum_{n\geqslant 0}(n+\tfrac{1}{2})t_{2n+1}R_n$$

$$\mathcal{S} := -\tfrac{1}{4}p + \sum_{n\geqslant 0}(n+\tfrac{1}{2})t_{2n+1}R_{n+1}$$

then we have

Theorem 2.1. A solution p of the potential KdV hierarchy is self-similar under the vector field X if and only if it satisfies the generalized string equation

$$\sum_{n\geqslant 0}\theta_{2n+1}R_{n+1} + v\mathcal{R} + w\mathcal{S} = 0. \tag{2.4}$$

Observe that when $w = 0$ the equation above depends only on u, thus given any self-similar solution p then $p + c$ is a self-similar solution as well. Notice also the relation [10]

$$\partial_1\mathcal{S} = (1/4\partial_1^3 + u\partial_1 + 1/2(\partial_1 u))\mathcal{R}$$

thus if p is Galilean self-similar, $\mathcal{R} = 0$, $\partial_1 \mathcal{S} = 0$ u is scaling self-similar, and there exists a corresponding scaling self-similar p.

The study of stationary manifolds for the KdV equation is just the study of solutions of the hierarchy which are invariant under certain translations [8]. The translational self-similarity condition ($v = w = 0$) is represented by the condition $\boldsymbol{\theta} p = 0$ or the Novikov equation [23]

$$\sum_{n\geqslant 0} \theta_{2n+1} R_{n+1} = 0.$$

This determines the finite gap solutions of the hierarchy. A Galilean self-similarity solution of the hierarchy ($\boldsymbol{\theta} = 0$, $w = 0$) is a solution of the string equation of the two-dimensional quantum gravity [7, 4, 13] $\mathcal{R} = 0$. In the scaling case ($\boldsymbol{\theta} = 0$, $v = 0$) one obtains $\mathcal{S} = 0$, a string equation that contains the one proposed recently in [6, 16, 3] as an alternative and non-perturbative approach to two-dimensional quantum gravity. In [5, 17] it is considered a combination of Galilean and scaling self-similarity, having the Galilean contribution the interpretation of a boundary cosmological term.

The general self-similarity condition can be reformulated using the zero-curvature condition for the hierarchy. We define the outer derivative

$$\delta := \mathbf{P}(\lambda)\frac{\mathrm{d}}{\mathrm{d}\lambda} + \frac{w}{4}\mathrm{ad}\, H \tag{2.5}$$

where $\mathbf{P}(\lambda) := v + w\lambda$, we have used the standard Cartan–Weyl basis $\{E, H, F\}$ for $sl(2, \mathbb{C})$. and

$$M := \langle \omega_+, X\rangle + \tfrac{1}{2} v t_1 F. \tag{2.6}$$

Here $\langle\cdot,\cdot\rangle$ is the standard pairing between 1-forms and vector fields. Then one has:

Theorem 2.2. The zero-curvature type condition

$$[d - \omega_+, \delta - M] = 0 \tag{2.7}$$

is equivalent to the generalized string equation (2.4).

Proof. This follows from the infinitesimal self-similarity condition

$$\delta\omega_+ = (vt_1 \mathrm{ad}\, F + L_X)\omega_+ + \tfrac{1}{2} vF\,\mathrm{d}t_1$$

where L_X denotes the Lie derivative along the vector field X. But

$$L_X\omega_+ = (i_X d + d i_X)\omega_+$$

and recalling the zero-curvature condition for ω_+, we obtain the desired result. □

When we are interested in translational self-similarity, $v = w = 0$, which leads to the finite gap solutions [8] to the KdV hierarchy. When v or w are not zero we are dealing with the so called quantization of finite gap potentials [24]. This formulation is closely related to the isomonodromic deformation approach [21].

3. The Grassmannian model and the string equations

In this section we use the Grassmannian model for the potential KdV flows in order to characterize geometrically the string equations for the self-similar solutions of the potential KdV hierarchy.

Recall that ω_+ defines a 1-form with values in the loop algebra $Lsl(2,\mathbb{C})$ of smooth maps from the circle $S^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ to the simple Lie algebra $sl(2,\mathbb{C})$. We also define an infinite set of commuting flows in the corresponding loop group $LSL(2,\mathbb{C})$

$$\psi(t,\lambda) := \sigma(t,\lambda)\cdot g(\lambda)$$

where g is the initial condition and

$$\sigma(t,\lambda) := \exp\left(\sum_{n\geqslant 0} t_{2n+1}\lambda^n J(\lambda)\right)$$

with $J(\lambda) := \lambda F + E$. Notice the role played by the Virasoro algebra and the Airy functions in this set of commuting flows. Define the outer derivations

$$L_n = \lambda^{n+1}\frac{\mathrm{d}}{\mathrm{d}\lambda} \qquad n = -1, 0, 1, \ldots$$

and

$$\mathcal{V}(t) := \sum_{n\geqslant 0} t_{2n+1} L_{n-1}.$$

Then $J = L_{-1}a\cdot a^{-1}$ where

$$a(\lambda) := \sqrt{\pi}\begin{pmatrix} \mathrm{Ai}(\lambda) & \mathrm{Bi}(\lambda) \\ \mathrm{Ai}'(\lambda) & \mathrm{Bi}'(\lambda)\end{pmatrix}\cdot a_0 \qquad a_0 \in SL(2,\mathbb{C})$$

and Ai, Bi are the standard Airy functions [22, 29]. Finally, one has the expression

$$\sigma(t) = \exp(\mathcal{V}(t)a\cdot a^{-1}).$$

Denote by $L^+SL(2,\mathbb{C})$ those loops which have a holomorphic extension to the interior of S^1, and by $L_1^-SL(2,\mathbb{C})$ those which extend to the exterior of the circle and are normalized by the identity at ∞. Given an element ψ of the loop group it belongs to the big cell (an open subset dense in the identity component of the loop group) if there exists a unique Birkhoff factorization as [26]

$$\psi = \psi_-^{-1}\cdot\psi_+ \tag{3.1}$$

where $\psi_- \in L_1^-SL(2,\mathbb{C})$ and $\psi_+ \in L^+SL(2,\mathbb{C})$. The solution to this factorization problem for $\psi(t)$ is deeply connected with the potential KdV hierarchy. The element ψ_- can be parametrized by a function p and its ∂_1 derivatives in such a way that ψ_- is a solution to the Birkhoff factorization problem if and only if p is a solution to the potential KdV hierarchy, therefore

$$\omega_+ := \mathrm{d}\psi_+\cdot\psi_+^{-1} = P_+\mathrm{Ad}\,\psi_-\left(\sum_{n\geqslant 0}\lambda^n J(\lambda)\,\mathrm{d}t_{2n+1}\right) \tag{3.2}$$

is the zero-curvature 1-form for the potential KdV equation [14]. Here $\mathrm{id} = P_+ + P_-$ is the resolution of the identity related to the splitting

$$Lsl(2, \mathbb{C}) = L^+sl(2, \mathbb{C}) \oplus L_1^-sl(2, \mathbb{C}).$$

One can conclude from these considerations that the projection of the commuting flows $\psi(t)$ on the Grassmannian manifold [28, 26]

$$LSL(2, \mathbb{C})/L^+SL(2, \mathbb{C}) \cong \mathrm{Gr}_\infty^{(2)}$$

is described by the potential KdV hierarchy and, consequently, by the KdV hierarchy.

We must remark that g determines a point in the Grassmannian up to the gauge freedom $g \mapsto g \cdot h$, where $h \in L^+SL(2, \mathbb{C})$. A solution of the potential KdV hierarchy does not change when $g(\lambda) \mapsto \exp(\beta(\lambda)J(\lambda)) \cdot g(\lambda)$ if $\exp(\beta J) \in L_1^-SL(2, \mathbb{C})$. If g is the initial condition associated with the solution p then $g_c(\lambda) = \exp(-c\lambda^{-1}J(\lambda)) \cdot g(\lambda)$ is the initial condition corresponding to the solution $p + c$; in fact we have $\psi_+ \mapsto \exp(-cF) \cdot \psi_+$. We can say that the *moduli* space for the KdV hierarchy contains the double coset space

$$\mathcal{M} := \Gamma_-\backslash LSL(2, \mathbb{C})/L^+SL(2, \mathbb{C})$$

where Γ_- is the Abelian subgroup with Lie algebra $\mathbb{C}\{\lambda^n J(\lambda)\}_{n<0}$.

Let us now try to find for which initial conditions g one gets self-similar solutions, i.e. subspaces in the Grassmannian that are connected to self-similar solutions of the potential KdV hierarchy. Recall that we have the derivation $\delta \in \mathrm{Der}\, L^+sl(2, \mathbb{C})$ defined in (2.5) and the vector $M(t) \in L^+sl(2, \mathbb{C})$ defined in (2.6). We denote

$$\theta(\lambda) := \sum_{n\geqslant 0} \theta_{2n+1}\lambda^n$$

then one has:

Theorem 3.1. If the initial condition g satisfies the equation

$$\delta g \cdot g^{-1} + \mathrm{Ad}\, gK = \theta J - \frac{v}{4\lambda} H \tag{3.3}$$

for $K \in L^+sl(2, \mathbb{C})$, then the corresponding solution to the potential KdV hierarchy satisfies the generalized string equation (2.4).

Proof. For $\omega_+ = \mathrm{d}\psi_+ \cdot \psi_+^{-1}$ we observe that the equation (2.7) holds if and only if

$$M = \delta\psi_+ \cdot \psi_+^{-1} + \mathrm{Ad}\, \psi_+ K \tag{3.4}$$

for some $K \in L^+sl(2, \mathbb{C})$. This, together with the factorization problem (3.1), implies the relation

$$M = \delta\psi_- \cdot \psi_-^{-1} + \mathrm{Ad}\, \psi_-(\delta\sigma \cdot \sigma^{-1} + \mathrm{Ad}\, \sigma(\delta g \cdot g^{-1} + \mathrm{Ad}\, gK)).$$

Now, $M(t) \in L^+sl(2, \mathbb{C})$ and equation (3.3) gives

$$M = P_+ \mathrm{Ad}\, \psi_- \left(\delta\sigma \cdot \sigma^{-1} + \mathrm{Ad}\, \sigma \left(\theta J - \frac{v}{4\lambda} H\right)\right).$$

One can easily compute

$$\delta\sigma\cdot\sigma^{-1}=\sum_{n\geqslant 0}\frac{t^n}{(n+1)!}(\operatorname{ad}J)^n\delta(tJ)\qquad \operatorname{Ad}\sigma X=\sum_{n\geqslant 0}\frac{t^n}{n!}(\operatorname{ad}J)^nX$$

where $t(\lambda)=\sum_n t_{2n+1}\lambda^n$. Thus one arrives at the expression

$$M=P_+\operatorname{Ad}\psi_-\left(\theta J+\sum_{n\geqslant 0}\frac{t^n}{(n+1)!}(\operatorname{ad}J)^n\left(\delta(tJ)+t\left[J,-\frac{v}{4\lambda}H\right]\right)\right)$$

but

$$\left(\delta(tJ)+t\left[J,-\frac{v}{4\lambda}H\right]\right)(\lambda)=\mathbf{P}(\lambda)\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}+\frac{t}{2\lambda}\right)(\lambda)J(\lambda)=\mathbf{P}(\lambda)\sum_{n\geqslant 0}(n+\tfrac{1}{2})t_{2n+1}\lambda^{n-1}J(\lambda)$$

and we deduce that

$$M=P_+\operatorname{Ad}\psi_-\left(\theta(\lambda)+\mathbf{P}(\lambda)\sum_{n\geqslant 0}(n+\tfrac{1}{2})t_{2n+1}\lambda^{n-1}\right)J(\lambda).$$

Taking into account equation (3.2) we recover (2.6) and therefore the generalized string equation is satisfied. □

Equation (3.3) admits the equivalent formulation

$$Ag=g\cdot k \tag{3.5}$$

where

$$A:=\mathbf{P}(\lambda)\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}+\frac{1}{4\lambda}H\right)-\theta(\lambda)J(\lambda)\qquad\text{and}\qquad k(\lambda):=\tfrac{1}{4}wH-K(\lambda).$$

We write

$$g=\begin{pmatrix}\varphi_1 & \tilde{\varphi}_1\\ \varphi_2 & \tilde{\varphi}_2\end{pmatrix}$$

with $\varphi_1\tilde{\varphi}_2-\tilde{\varphi}_1\varphi_2=1$, and introduce the notation

$$\Phi=\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}\qquad \tilde{\Phi}=\begin{pmatrix}\tilde{\varphi}_1\\ \tilde{\varphi}_2\end{pmatrix}.$$

Define also the map [26, 28] $\Phi\mapsto\varphi:=T\Phi$ where $(T\Phi)(\lambda):=\lambda\varphi_1(\lambda^2)+\varphi_2(\lambda^2)$.

To characterize the subspace for which the associated solution of the KdV hierarchy satisfies the generalized string equation, denote

$$\xi:=\begin{pmatrix}\varphi\\ \tilde{\varphi}\end{pmatrix}$$

and define

$$\mathcal{A}:=T\circ A\circ T^{-1}=\mathbf{P}(\lambda^2)\left(\frac{1}{2\lambda}\frac{\mathrm{d}}{\mathrm{d}\lambda}-\frac{1}{4\lambda^2}\right)-\lambda\theta(\lambda^2)$$

then we have:

Proposition 3.1. Any solution of the system of first-order ordinary differential equations

$$\mathcal{A}\xi(\lambda) = k^t(\lambda^2)\xi(\lambda) \tag{3.6}$$

gives the subspace

$$W := \mathbb{C}\{\lambda^{2n}\varphi, \lambda^{2n}\tilde{\varphi}\}_{n\geqslant 0}$$

which is the point in the Grassmannian related to the solution of the generalized string equation (2.4). This subspace is characterized by $\mathcal{A}W \subset W$. The condition $\det g = 1$ gives the constraint

$$\varphi(\lambda)\tilde{\varphi}(-\lambda) - \varphi(-\lambda)\tilde{\varphi}(\lambda) = 2\lambda.$$

Notice that this subspace does not belong generically to the Segal–Wilson Grassmannian $\mathrm{Gr}_{\infty}^{(2)}$. Another possibility is represented by the Sato Grassmannian, but again not always this subspace belongs to this manifold.

4. Description of self-similar solutions in the Sato Grassmannian

In this section we find the points in the Sato Grassmannian corresponding to self-similar solutions of the potential KdV hierachy. The constructions of the Grassmannian given by Sato [27] and Segal–Wilson [28] differs mainly in the choice of the underling space $\mathcal{H}$. In the Segal–Wilson Grassmannian we are dealing with the Hilbert space $\mathcal{H} = L^2(S^1, \mathbb{C})$ which in the Sato case is replaced by the space of formal series $\mathbb{C}[[\lambda^{-1}, \lambda]$. The statements of the previous section which are rigorous in the Segal–Wilson case, can be extended to the Sato frame if the formal group $L_{\Gamma}^{-}SL(2,\mathbb{C})$ is considered only when acting by its adjoint action or by gauge transformations in the formal Lie algebra $sl(2,\mathbb{C})[[\lambda^{-1}, \lambda]$. In this context equations (2.7), (3.4) and (3.3) still hold.

The subspace in the Sato Grassmannian corresponding to a self-similar solution is fixed by an initial condition g in the formal *moduli* $\mathcal{M}$. If this is the case equation (3.5) gives us the canonical structure of k and therefore, we get the corresponding point in the Sato Grassmannian in a canonical way. Notice that for each equivalence class in $\mathcal{M}$ an element g can be taken such that $\ln g \in sl(2,\mathbb{C})[[\lambda^{-1}])$.

Since $\sigma|_{t=0} = \mathrm{id}$ it follows from (3.1) that $\psi_+|_{t=0} = \mathrm{id}$ and equation (3.4) gives $K = M|_{t=0}$. But, from (2.6) we have $K = \langle\omega_+|_{t=0}, \theta\rangle$ where we have taken into account that $X|_{t=0} = \theta$ and we obtain the expression for k,

$$k = \tfrac{1}{4}wH - \langle\omega_+|_{t=0}, \theta\rangle.$$

Recalling the formula (2.2) we write

$$k = \tfrac{1}{4}wH - \langle\mathrm{Ad}\,\phi_0\chi|_{t=0}, \theta\rangle - \theta p|_{t=0}F$$

where $\phi_0 = \phi|_{t=0}$. Noting that

$$\tfrac{1}{4}wH = \mathrm{Ad}\,\phi_0(\tfrac{1}{4}wH) - \tfrac{1}{2}wp_0F$$

from the string equation (2.3) one concludes

$$k = \mathrm{Ad}\,\phi_0(\tfrac{1}{4}wH - \langle\chi|_{t=0}, \theta\rangle).$$

Obviously the subspace W does not change when we remove the adjoint action of ϕ_0, this gives

$$k = \tfrac{1}{4}wH - \langle \chi|_{t=0}, \theta\rangle. \tag{4.1}$$

Theorem 4.1. The point W in the Grassmannian corresponding to a self-similar solution of the potential KdV hierarchy is given by the solutions of

$$\left\{\mathbf{P}(\lambda^2)\left(\frac{1}{2\lambda}\frac{\mathrm{d}}{\mathrm{d}\lambda} - \frac{1}{4\lambda^2}\right) - \lambda\theta(\lambda^2) - \frac{w}{4}H + \sum_{n\geqslant 0}\theta_{2n+1}L^t_{2n+1}|_{t=0}(\lambda^2)\right\}\xi(\lambda) = 0$$

having the asymptotic expansion

$$\xi(\lambda) \sim \begin{pmatrix} \lambda + \varphi_{11}\lambda^{-1} + \varphi_{21}\lambda^{-2} + \cdots \\ 1 + \tilde{\varphi}_{11}\lambda^{-1} + \tilde{\varphi}_{21}\lambda^{-2} + \cdots \end{pmatrix} \qquad \lambda \to \infty.$$

Notice that when $\theta(\lambda) = a(N+1/2)\mathbf{P}(\lambda)\lambda^{N-1}$ the translation term in the string equation is removed if we transform the time coordinates as follows: $t_{2n+1} \mapsto t_{2n+1} + a\delta_{nN}$. Only when $v = 0$ (scaling case) can we translate t_1 with this procedure. In this sense translation, scaling and Galilean self-similarity can model, in translated time coordinates, scaling and Galilean self-similarity.

4.1. Scaling self-similarity

Let us consider the scaling case alone, $v = 0$, $w = 1$. Because the solutions are singular when $t = 0$ we consider the vector field $X = \varsigma - \partial_1$ for which $\theta(\lambda) = -1$ and $\mathbf{P}(\lambda) = \lambda$. Observe that in the coordinates $\tilde{t}_1, t_3, t_5, \ldots$ with $\tilde{t}_1 = t_1 - 2$ we are dealing with scaling self-similar solutions. We have

$$k^t(\lambda^2) = \begin{pmatrix} 1/4 & \lambda^2 - u_0 \\ 1 & -1/4 \end{pmatrix}$$

where $u_0 = u|_{t=0}$. equation (3.6) with

$$\mathcal{A} = \lambda^2\left(\frac{1}{2\lambda}\frac{\mathrm{d}}{\mathrm{d}\lambda} - \frac{1}{4\lambda^2}\right) + \lambda$$

is equivalent to

$$\varphi = \lambda\left(\tfrac{1}{2}\tilde{\varphi}' + \tilde{\varphi}\right) \qquad \text{and} \qquad \tilde{\varphi}'' + 4\tilde{\varphi}' + \frac{4u_0}{\lambda^2}\tilde{\varphi} = 0.$$

From now on we shall denote $f' = \mathrm{d}f/\mathrm{d}\lambda$. Introducing the asymptotic expansion

$$\tilde{\varphi} \sim 1 + \tilde{\phi}_1\lambda^{-1} + \tilde{\phi}_2\lambda^{-2} + \cdots \qquad \lambda \to \infty$$

we find the recurrence relation

$$\tilde{\phi}_{n+1} = \left(\frac{n}{4} + \frac{u_0}{n+1}\right)\tilde{\phi}_n \qquad \tilde{\phi}_0 = 1.$$

This expansion converges only when it is a finite series, which happens if and only if

$$u_0 = -\tfrac{1}{4}m(m+1) \tag{4.2}$$

for some $m \in \mathbb{N} \cup \{0\}$, and then

$$\tilde{\phi}_n = 0 \qquad n > m.$$

When (4.2) is satisfied we are dealing with a rational solution of the KdV hierarchy that for $t = \{t_1, 0, 0, \ldots\}$ is of the form $u = -m(m+1)/(t_1 - 2)^2$. These are the well known rational solutions of the KdV hierarchy, that vanish at $t_1 = \infty$, analysed by Adler and Moser [2]. They have already noted the scaling properties of these solutions for which the corresponding subspace not only belongs to the Sato Grassmannian but also to the Segal–Wilson Grassmannian $\mathrm{Gr}_0^{(2)}$. For an arbitrary u_0 we have a point in the Sato Grassmannian, so there is a one-dimensional complex curve in this space giving scaling self-similar solutions. Define $\nu = \frac{1}{2}\sqrt{1 - 16u_0}$ then we have

$$\tilde{\varphi}(\lambda) \sim \mathrm{i}\sqrt{\frac{4\lambda}{\pi}}\mathrm{e}^{-2\lambda}K_\nu(-2\lambda) \sim \sum_{n\geqslant 0}(-1)^n \frac{\Gamma(\nu + n + \frac{1}{2})}{4^n n!\Gamma(\nu - n + \frac{1}{2})}\lambda^{-n} \qquad \lambda \to \infty$$

here K_ν is the Macdonald function [11, 22]. Observe that if (4.2) is satisfied then $\nu = m + 1/2$, and $\tilde{\varphi}$ is the following polynomial in λ^{-1}

$$\tilde{\varphi}(\lambda) = \lambda^{m+1}\mathrm{e}^{-2\lambda}\left(\frac{1}{2\lambda}\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^{m+1}\mathrm{e}^{2\lambda}.$$

More generally, one can consider $\theta(\lambda)$ as a polynomial of degree M, then $k(\lambda)$ depends on

$$\{R_{n,0}, \dot{R}_{n,0}, \ddot{R}_{n,0}\}_{n=1}^{M}$$

where we denote by $\dot{f} = \partial_1 f$. These constants are not independent, in fact we have the relations [10]

$$R_{n+1,0} = 2\sum_{m=0}^{n-1} R_{m,0}\ddot{R}_{n-m,0} - \sum_{m=1}^{n-1}\dot{R}_{m,0}\dot{R}_{n-m,0} + 4u_0\sum_{m=0}^{n} R_{m,0}R_{n-m,0} - 4\sum_{m=1}^{n} R_{m,0}R_{n-m+1,0}.$$

Since these are all the constraints that must be satisfied by the constants we conclude that k is parametrized by a $(2M+1)$-dimensional algebraic variety $\Sigma_\theta \subset \mathbb{C}^{3M}$. For each point in this variety we have a subspace in the Sato Grassmannian $\mathrm{Gr}^{(2)}$, this map gives an inclusion $\Sigma_\theta \hookrightarrow \mathrm{Gr}^{(2)}$. This $(2M+1)$–dimensional surface intersects the Segal–Wilson Grassmannian $\mathrm{Gr}_0^{(2)}$ in a discrete set, that can be labelled by $\mathbb{N}$, in fact each point in this intersection set corresponds to an Adler–Moser rational solution.

When $0 = t_5 = t_7 = \ldots$ we are dealing with the KdV equation, and the scaling self-similarity condition leads to the Painlevé II equation. Therefore, the polynomial θ is of the form $\theta(\lambda) = \theta_1 + \theta_3\lambda$. Moreover in the coordinates $\tilde{t}_1 = t_1 + 2\theta_1$, $\tilde{t}_3 = t_3 + \frac{2}{3}\theta_3$ we have scaling self-similar solutions of the KdV equation which are well defined in a neighbourhood of $\tilde{t}_1 = \tilde{t}_3 = 0$. In the Sato Grassmannian we have a three-dimensional surface giving us self-similar solutions as we said before. Let us note that in [9] a family of self-similar solutions also depending on three parameters appears (see [1] and references therein).

4.2. Galilean self-similarity

Suppose now $w = 0$, that is, we consider translations and Galilean self-similarity. As before we set $X = \gamma - \partial_1$, ($\theta(\lambda) = -1$) but now $\mathbf{P}(\lambda) = 1$. Observe that in the coordinates $t_1, \tilde{t}_3, t_5, \ldots$ with $\tilde{t}_3 = t_1 - \frac{2}{3}$ we are considering Galilean self-similar solutions. We get in this case

$$k^t(\lambda^2) = \begin{pmatrix} 0 & \lambda^2 - u_0 \\ 1 & 0 \end{pmatrix}$$

but the string equation (2.4) implies $R_{1,0} = 0$, and so $u_0 = 0$. Hence

$$k^t(\lambda^2) = \begin{pmatrix} 0 & \lambda^2 \\ 1 & 0 \end{pmatrix}$$

and the operator $\mathcal{A}$ becomes

$$\mathcal{A} = \frac{1}{2\lambda}\frac{\mathrm{d}}{\mathrm{d}\lambda} - \frac{1}{4\lambda^2} + \lambda.$$

The equation (3.6) determining the subspace W can be transformed into a second-order ordinary differential equation:

$$(\mathcal{A}^2 - \lambda^2)\tilde{\varphi} = 0$$

and the function φ is given by $\varphi = \mathcal{A}\tilde{\varphi}$. Introducing the asymptotic expansion

$$\tilde{\varphi}(\lambda) \sim \sum_{n\geqslant 0} \tilde{\phi}_n \lambda^{-n} \qquad \lambda \to \infty$$

in the equation

$$\tilde{\varphi}'' + 2\frac{2\lambda^3 - 1}{\lambda}\tilde{\varphi}' + \frac{5}{4\lambda^2}\tilde{\varphi} = 0$$

one finds the recurrence relation

$$\tilde{\phi}_{n+3} = \left(\frac{n}{4} + \frac{5}{16(n+3)}\right)\tilde{\phi}_n \qquad \tilde{\phi}_0 = 1 \qquad \tilde{\phi}_1 = \tilde{\phi}_2 = 0.$$

One can also write

$$\tilde{\varphi}(\lambda) \sim \mathrm{i}\sqrt{\frac{4\lambda^3}{3\pi}}\mathrm{e}^{-2\lambda^3/3} K_{1/3}(-\tfrac{2}{3}\lambda^3) \sim \sum_{n\geqslant 0}(-1)^n \frac{3^n\Gamma(n+\frac{5}{6})}{4^n n!\Gamma(-n+\frac{5}{6})}\lambda^{-n} \qquad \lambda \to \infty.$$

From these formulae it follows that there is only one point in the Sato Grassmannian associated with a self-similar solution of the hierarchy for $X = \gamma - \partial_1$; note that this solution corresponds to a Galilean self-similar solution defined in the neighbourhood of $t_3 = 2/3$. This was found by Kac and Schwarz in [18], and apparently one can conclude that there is a unique point, in the Sato Grassmannian, corresponding to the solution of the string equation of the double scaling limit of the Hermitian matrix model. As we shall see one can find, for example, a $2M$-dimensional surface in the Sato Grassmannian corresponding to Galilean self-similar solutions (this result was found in [21] in terms of some Stokes matrices) and when $M = 0$ we recover the result of [18]. Suppose that $\theta(\lambda)$ is a polynomial of degree M, then the matrix k depends on the algebraic variety Σ_θ defined previously. But now there is an additional constraint to be satisfied by the parameters of k. The string equation (2.4) when evaluated at $t = 0$ gives

$$\sum_{n=0}^{M} \theta_{2n+1} R_{n+1,0} = 0$$

which fixes an $2M$-dimensional algebraic subvariety $\tilde{\Sigma}_\theta$. Then, equation (3.6) gives a point in the Sato Grassmannian $\mathrm{Gr}^{(2)}$ and in this way $\tilde{\Sigma}_\theta$ is included in $\mathrm{Gr}^{(2)}$. Therefore we have for each polynomial θ of degree M a $2M$-dimensional surface in the Grassmannian such that each point in it is associated with a self-similar (under $\theta + \gamma$) solution of the KdV hierarchy. Notice that it does not intersect the Segal–Wilson Grassmannian. Observe also that when $M = 0$ we obtain a zero-dimensional space, a point in fact.

Let us analyse the case $\theta(\lambda) = -\lambda$, which corresponds to a self-similar solution under $\gamma - \partial_3$. In the coordinates $t_1, t_3, \tilde{t}_5, t_7, \ldots$, with $\tilde{t}_5 = t_5 - \frac{2}{5}$ we have Galilean self-similar solutions. Now $M = 1$, and the dimension of our variety $\tilde{\Sigma}$ is 2. The matrix k^t is

$$k^t(\lambda) = \begin{pmatrix} -\frac{1}{2}\dot{u}_0 & \lambda^2 - \frac{1}{2}u_0\lambda - \frac{1}{4}(\ddot{u}_0 + 2u_0^2) \\ \lambda + \frac{1}{2}u_0 & \frac{1}{2}\dot{u}_0 \end{pmatrix}$$

but the string equation implies $R_{2,0} = 0$ so that

$$\ddot{u}_0 = -3u_0^2 \qquad \text{and} \qquad k^t(\lambda) = \begin{pmatrix} -\frac{1}{2}\dot{u}_0 & \lambda^2 - \frac{1}{2}u_0\lambda + \frac{1}{4}u_0^2 \\ \lambda + \frac{1}{4}u_0 & \frac{1}{2}\dot{u}_0 \end{pmatrix}.$$

With this expression and

$$\mathcal{A} = \frac{1}{2\lambda}\frac{\mathrm{d}}{\mathrm{d}\lambda} - \frac{1}{4\lambda^2} + \lambda^3$$

one obtains from equation (3.6) a second-order ordinary differential equation for $\tilde{\varphi}$

$$(\mathcal{A} + \tfrac{1}{2}\dot{u}_0)[1/(\lambda^2 + \tfrac{1}{2}u_0)](\mathcal{A} - \tfrac{1}{2}\dot{u}_0)\tilde{\varphi} = (\lambda^4 - \tfrac{1}{2}u_0\lambda + \tfrac{1}{4}u_0^2)\tilde{\varphi}$$

and the relation

$$(\mathcal{A} - \tfrac{1}{2}\dot{u}_0)\tilde{\varphi} = (\lambda^2 + \tfrac{1}{2}u_0)\varphi.$$

From these two equations one finds asympotic expansions

$$\varphi \sim \lambda + \phi_1\lambda^{-1} + \cdots \qquad \lambda \to \infty$$
$$\tilde{\varphi} \sim 1 + \tilde{\phi}_1\lambda^{-1} + \tilde{\phi}_2\lambda^{-2} + \cdots \qquad \lambda \to \infty$$

where the coefficients $\phi_n, \tilde{\phi}_n$ are obtain from a recurrence relation and depends on the initial data $u_0, \dot{u}_0$. In this case $\tilde{\Sigma}$ is a quadric in $\mathbb{C}^3$.

4.3. Galilean and scaling self-similarity

Finally, we suppose that $v, w \neq 0$ and let θ be a polynomial of degree M. The matrix k' is as before, but now $w \neq 0$ and the string equation does not give any extra condition, and it is parametrized by a $(2M+1)$-dimensional algebraic variety Σ_θ. Equation (3.6) gives a point in the Sato Grassmannian for each point in this algebraic variety. That is, there is a $(2M+1)$-dimensional surface in $\mathrm{Gr}^{(2)}$ so that its points are associated with self-similar solutions under the vector field X. The intersection of this set with the Segal–Wilson Grassmannian is the empty set and only when $v=0$, there is an intersection as we said before. Galilean self-similarity implies that we are working out of the Segal–Wilson frame.

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